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D3.4 Master equation for the dynamics of rare-earth ion coupled to mechanical vibrations

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Deliverable description

This deliverable consists in a report on master equation approach for describing the dynamics of rare earth ions coupled to mechanical vibrations in relation with MS4.

Introduction and context

In a previous article [1] we demonstrated that structured hole burning can prepare a cantilever such that light transmitted through the spectral hole acquires a phase shift proportional to the bending of the cantilever. We discussed the ability to detect thermal and vacuum Brownian motion of the cantilever via the noise spectrum of the transmitted radiation.

The next steps consist of determining how to exploit the coupling between rare-earth ions and resonator in order to take advantage of the quantum nature of the resonator, either for creating non-classical states (D3.7) or use for sensing purposes (D3.9).

In this context, we wish to study the measurement back action on the quantum state of the cantilever due to continuous homodyne detection. The information obtained about the cantilever motion reduces its uncertainty and is fully equivalent to a cooling mechanism: While we do not on average extract energy from the oscillator, we obtain a narrow probability distribution around a random (but known) displacement in the position-momentum phase space. This displacement constitutes a reference frame in which the energy (and temperature) is reduced, and if needed, it can be either removed by application of a force on the cantilever, or its known value can be merely employed in applications such as motion and force sensing. We have established a Gaussian covariance matrix description of the dynamics which is exact as the interactions, damping and continuous probing maintain the Gaussian state property.

Theoretical Approach

A transparent cantilever is probed by a coherent laser beam of light, which undergoes a phase shift $\Delta\Phi = k\, x_m$, where x_m denotes the effective coordinate of one of the vibrational modes of the mechanical oscillator, and k is a constant. In the following we assume that x_m is dimensionless, i.e., the position coordinate is given in units of its rms uncertainty in the ground state x_0. We will use this phase shift in order to monitor the motion of the resonator.

In ref. [1], we identified an oscillator mode at ω=890 kHz with effective mass of m=1.1 10^{-11} kg, causing a phase shift of 0.2 mrad for a bending of 0.4 pm, equivalent to a phase shift of 5.7 μrad for a bending of 1.3 fm (the oscillator ground state width) and k= 5.7 μrad.

The details of the calculations can be found in the appendix, here we will focus on the main results.

Equations for the co-variance matrix

The Gaussian covariance matrix solves a non-linear, so-called, Riccati equation, and we obtain for the oscillator part of the covariance matrix the following deterministic component equations:

$$\frac{da_{11}}{dt} = -\eta\kappa^2 a_{11}^2 + \omega(a_{21} + a_{12}) - \gamma(a_{11} - (2\overline{n} + 1))$$
$$\frac{da_{12}}{dt} = -\eta\kappa^2 a_{11} a_{12} - \omega(a_{11} - a_{22}) - \gamma a_{12}$$
$$\frac{da_{21}}{dt} = -\eta\kappa^2 a_{11} a_{21} - \omega(a_{11} - a_{22}) - \gamma a_{21}$$
$$\frac{da_{22}}{dt} = \kappa^2 - \eta\kappa^2 a_{12} a_{21} - \omega(a_{21} + a_{12}) - \gamma(a_{22} - (2\overline{n} + 1))$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

represents the oscillator position and momentum variances and co-variances. We have also defined $\kappa^2 = k^2\Phi$ with Φ the photon flux, and η is the photon detection efficiency, and γ is the mechanical decay rate. Finally, n represents the temperature of the thermal bath surrounding the system, driving the oscillator towards a thermal state with mean excitation n.

Interpretation of equations

By solving the above equations, we obtain a_{11} which corresponds to twice the position variance of the resonator. By continuously probing the resonator by monitoring the phase of a transmitted laser beam, the figure shows that it is possible to reach a value for a_{11} close to the variance corresponding to the zero-point motion of the resonator in a few microseconds (see figure 1) which corresponds to possessing the maximum information of the resonator position.

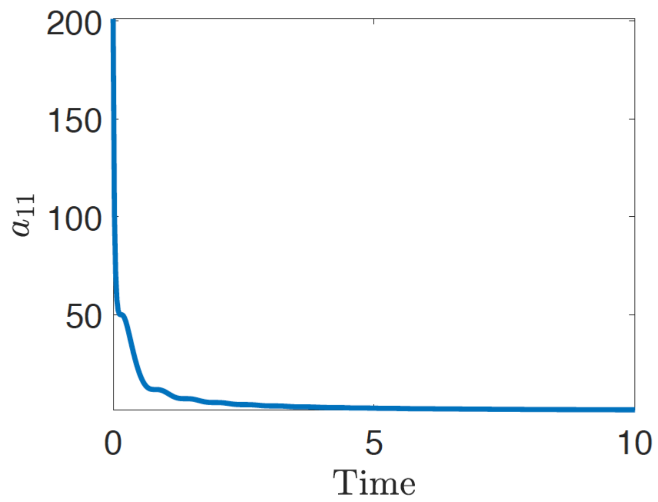

Figure 1 Twice the position variance for a cantilever subject to optical probing. The cantilever parameters are given in the beginning of the report, and we assume κ^2 = 0.2 MHz, η = 1, bath excitation n = 100, and bath coupling γ = 50 Hz. The time is in micro-seconds.

Note that the unobserved oscillator momentum undergoes an increasing variance due to the interaction with the probe field (first term in the rate equation for a_{22}) - this diffusive heating is the back action on the mechanical oscillator of the photon number fluctuation in the incident state.

From the above equations, it is also possible to simulate the position of the resonator as a function of time as shown in figure 2.

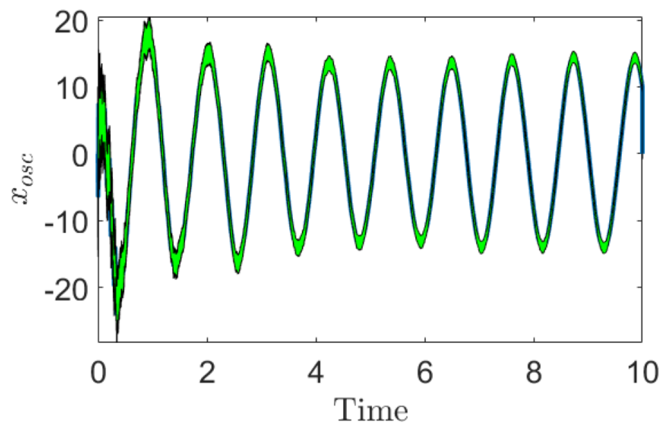

Figure 2 Simulated evolution of the position of the cantilever. The system acquires a well-defined phase and amplitude, depending on the early measurement record. The width of the coloured area is obtained from a_{11} in the previous figure (fig 1) and the parameters used in the simulation are the same. Time is in units of μs.

Conclusion

We have in the framework of the NanOQTech project derived a Gaussian state formalism that accounts for the evolution of a mechanical oscillator subject to continuous homodyne probing. The measurement outcome is stochastic, and the measurement back action entails a displacement of the oscillator, which one must know to benefit from the significantly reduced variance of the inferred position and momentum of the cantilever. The increased purity of the quantum state accompanies a reduced entropy, and we shall refer to the process as *measurement induced cooling*, as the residual energy of the system is mainly due to a precisely known oscillatory motion in phase space – a motion that can be arrested by application of a force, and that does not deteriorate the application of the cantilever for force or inertial sensing, as long as it is known.

In succession to the work reported here, we plan to

- extract analytical expression for the variances as well as values in limiting and optimal cases,
- develop and investigate the performance of feedback and sensing protocols,
- make a separate study of the so-called retrodicted state [2] of the system: What do we know at time T about the oscillator's position at the earlier time t, due to the measurements performed both until t and after t, and how can this benefit for motional sensing?

Bibliography

[1] K. Mølmer, Y. Le Coq and S. Seidelin, *Dispersive coupling between light and a rare-earth ion doped mechanical resonator*, Physical Review A **94**, 053804 (2016).

[2] Søren Gammelmark, Brian Julsgaard, and Klaus Mølmer, *Past Quantum States of a Monitored System*, Phys. Rev. Lett. **111**, 160401 (2013).

Appendix: Details on calculations concerning the conditional master equation for the motion of a continuously monitored cantilever

The physical system

A cantilever is probed by a coherent laser beam of light, which undergoes a phase shift $\Delta\phi = kx_m$, where x_m denotes the effective coordinate of one of the vibrational modes of the mechanical oscillator, and k is a constant. In the following we assume that x_m is dimensionless, i.e., the position coordinate is given in units of its rms uncertainty in the ground state x_0. In [1], we identified a $\omega \sim 890$ kHz mode for an oscillator mode with effective mass $m = 1.1 \cdot 10^{-11}$ kg, causing a phase shift of 0.2 mrad for a bending of 0.4 pm, equivalent to a phase shift 5.7μrad for a bending of $x_0 = 1.3$ fm, the oscillator ground state width ($k = 5.7\mu$rad).

The coherent beam of light can be thought of as a product state of segments of duration τ, each containing a coherent state with average photon number $n = \Phi\tau$. A Fock state $|n\rangle$ undergoes a quantum phase shift, $|n\rangle \to e^{-i\Delta\phi n}|n\rangle$. We shall assume that the field is in a coherent state with real argument $\alpha = \sqrt{n} = \sqrt{\Phi\tau}$, and write $\hat{a} =: \alpha + \delta\hat{a}$, such that

$$\begin{aligned} \hat{n} &= \hat{a}^\dagger\hat{a} = (\alpha + \delta\hat{a}^\dagger)(\alpha + \delta\hat{a}) \\ &\simeq \alpha^2 + \alpha(\hat{a} - \alpha) + \alpha(\hat{a}^\dagger - \alpha) = \alpha(\hat{a} + \hat{a}^\dagger) - \alpha^2 \\ &= \alpha x_{ph} - \alpha^2. \end{aligned} \tag{1}$$

This expression allows us to approximate the phase factor $e^{-i\Delta\phi n}$ by $e^{-i\kappa_\tau x_{ph} x_m}$, where $\kappa_\tau^2 \equiv \kappa^2\tau \equiv k^2\Phi\tau$.

Note that the exponential operator form reflects the unitary evolution of a field and a mechanical oscillator, with a coupling Hamiltonian $H = \kappa_\tau x_{ph} x_m/\tau$.

Evolution of the system; formalism

The evolution by the interaction Hamiltonian is given in both classical and quantum mechanics by a linear mapping of the position and momentum observables. In addition the oscillator Hamiltonian causes a phase space rotation at frequency ω of the oscillator coordinates, and for a short time interval τ, we get

$$\begin{pmatrix} x_m \\ p_m \\ x_{ph} \\ p_{ph} \end{pmatrix} \to \begin{pmatrix} 1 & \omega\tau & 0 & 0 \\ -\omega\tau & 1 & \kappa_\tau & 0 \\ 0 & 0 & 1 & 0 \\ \kappa_\tau & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_m \\ p_m \\ x_{ph} \\ p_{ph} \end{pmatrix}. \tag{2}$$

We now introduce the covariance matrix $\gamma_{ij} = 2Re(\langle(\hat{q}_i - q_i)(\hat{q}_j - q_j)\rangle$, where $\hat{q}_i$ denotes the four quadrature observables and q_i their expectation values. The state before passage of a segment of the coherent light beam is described by

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \tag{3}$$

where

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4}$$

represent the vacuum variances of the coherent state and its lack of prior correlation with the oscillator, and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{5}$$

represents the oscillator position and momentum variances and covariances.

Due to the interaction of the two systems, they become correlated as represented by the transformation

$$\gamma \to S\gamma S^T \tag{6}$$

where S denotes the 4x4 matrix in (2).

Note that each new segment of coherent light enters with the same B matrices and C matrices, while the mechanical resonator is described by the A, resulting from the previous interaction. If we were to just let the light escape to infinity, the oscillator would thus be described by the upper left block matrix A, which will be time dependent due to the continuous interaction with new coherent state segments of light.

We include the thermalization of the oscillator at (energy) rate γ to its environment, by the rate equations, $da_{ii}/dt = \gamma(2\overline{n} + 1) - \gamma a_{ii}$ for $i = 1, 2$, and $da_{ij}/dt = -\gamma a_{ij}$ for $i \neq j$. In the absence of other terms this would lead to a steady state of $a_{11} = a_{22} = 2Var(x_m) = 2Var(p_m) = 2\overline{n} + 1$, representing the familiar average $\frac{1}{2}\langle x_m^2 + p_m^2\rangle = (\overline{n} + \frac{1}{2})$ of the thermalized oscillator.

Now, in addition, we want to describe the effect of the homodyne measurement of the p_{ph} quadrature of the field segment after the interaction with the cantilever (containing the information about x_m, due to the phase rotation of the incident coherent field). We skip the technical and lengthy derivation here, but note that each segment that just leaves the cantilever and the cantilever itself are described by a joint Gaussian phase space distribution with a covariance matrix of the form (3), and a vector of the mean values of the four phase space variables in Eq.(2). The homodyne measurement of the field quadrature p_{ph} variable is predicted to follow a Gaussian probability distribution, but when the measurement reveals its value, we must evaluate the joint probability distribution of the field and cantilever at this observed value, and what remains is a (conditional) Gaussian phase space probability distribution for the cantilever state. This is nothing but a continuous version of Borns rule and the projection postulate acting on a composite system (which, in turn, is an implementation of Bayes' rule of conditional probabilities).

Evolution of the system; explicit equations and steady state

The Gaussian covariance matrix solves a non-linear, so-called, Riccati equation (the quadratic term a_{11} is not an error). Putting all terms together, assuming one segment after each other, and performing the derivative, $dx/dt = (x(t+\tau)-x(t))/\tau$, we get for the oscillator part of the covariance matrix the following deterministic component equations:

$$\begin{aligned}
\frac{da_{11}}{dt} &= -\eta\kappa^2 a_{11}^2 + \omega(a_{21}+a_{12}) - \gamma(a_{11}-(2\overline{n}+1)) \\
\frac{da_{12}}{dt} &= -\eta\kappa^2 a_{11}a_{12} - \omega(a_{11}-a_{22}) - \gamma a_{12} \\
\frac{da_{21}}{dt} &= -\eta\kappa^2 a_{11}a_{21} - \omega(a_{11}-a_{22}) - \gamma a_{21} \\
\frac{da_{22}}{dt} &= \kappa^2 - \eta\kappa^2 a_{12}a_{21} - \omega(a_{21}+a_{12}) - \gamma(a_{22}-(2\overline{n}+1)).
\end{aligned} \tag{7}$$

Note that the unobserved p_m undergoes an increasing variance due to the interaction with the probe field - this diffusive heating is the back action on the mechanical oscillator of the spread in x_{ph} of the incident state. If there was no free rotation (at ω), the effective anti-squeezing of p_m would be necessary to accompany the squeezing of x_m. Due to the rotation, however, we find a constant cooling of one and heating of the other degree of freedom and a constant mixing of the two, leading, ideally, to cooling of both. The parameter η denotes the propagation and detection efficiency (intensity), and putting $\eta = 0$ corresponds to detection, but still the p_m is heating up due to the interaction and entanglement with the probe field.

Together with the determistic change of the covariance matrix, the mean values of x_m and p_m develop non-zero mean values due to the measurements. Let dW denote the difference between the measured value and the expected mean value; dW is stochastic with variance $dW^2 = dt$, corresponding to detector shot noise, and its explicit variation leads to the update equation for the mean values:

$$\begin{aligned}
\langle x_m\rangle &\to \langle x_m\rangle + \sqrt{\eta}a_{11}\kappa dW \\
\langle p_m\rangle &\to \langle p_m\rangle + \sqrt{\eta}a_{21}\kappa dW
\end{aligned} \tag{8}$$

We can solve the equations with real experimental or simulated data, and in this way we can present the evolution of the motional state of the cantilever.

An example is given in Fig. 1 in the report showing the rapid decrease of the covariance matrix elements a_{11} (twice the variance of x_m) due to probing and Fig. 2 shows the conditional variation of the position quadrature (mean value and error interval).

[1] Klaus Mølmer, Yann Le Coq, and Signe Seidelin, *Dispersive coupling between light and a rare-earth ion doped mechanical resonator*, Phys. Rev. A **94**, 053804 (2016).